A-APR Egyptian Fractions II

Alignments to Content Standards: A-APR.D.6

Task

Ancient Egyptians used unit fractions, such as $\frac{1}{2}$ and $\frac{1}{3}$, to represent all other fractions. For example, they might express the number $\frac{3}{4}$ as $\frac{1}{2}+\frac{1}{4}$. The Egyptians did not use a given unit fraction more than once so they would not have written $\frac{3}{4}=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$.

a. How might the Egyptians have expressed the number $\frac{2}{7}$? What about $\frac{5}{13}$?

b. We will see how we can use identities between rational expressions to help in our understanding of Egyptian fractions. Verify the following identity for any $x>0$:

$$\frac{1}{x}=\frac{1}{x+1}+\frac{1}{x(x+1)}$$

c. Show that each unit fraction $\frac{1}{n}$, with $n \geq 2$, can be written as a sum of two or more different unit fractions.

d. Describe a procedure for writing any positive rational number, say, $\frac{p}{q}$ with $p, q>0$, as an Egyptian fraction.

IM Commentary

The purpose of this task is for students rewrite a simple rational expression and study the arithmetic of these expressions. Egyptian fractions provide an interesting context, both historically and mathematically, for students to use rational expressions. Analogous to the standard A-APR.4, in which students use polynomial identities to

describe numerical relationships, this task has students rewrite rational expressions (A-APR.D) in order to deduce information about rational *numbers*.

Specifically, students will work to rewrite fractions as a sum of distinct unit fractions, for which the identities of rational functions provided will be quite useful. In order to solve the problem, students will rewrite a simple rational expression and will need to study carefully the arithmetic of these unit fractions with a variable in the denominator. Depending on implementation (e.g., levels of scaffolding), the task poses reasonably high levels of cognitive demand and gives students the opportunity to model many of the Standards for Mathematical Practice. Consequently, teachers should be aware that a fully open-ended implementation of this task will require ample time and guidance. The remainder of this commentary describes some discussion of implementation.

Teachers may find wide divergence in approaches taken by students, e.g., coming up with the provided algebraic identities on their own (if they are initially withheld), or the largely equivalent "greedy algorithm" demonstrated in the solution, in which students simply choose the largest unit fraction under the given fraction (e.g., $\frac{1}{4}$ under $\frac{2}{7}$). As such, the teacher may wish to encourage strategy-sharing in group- or whole-class discussion, and/or give more examples for them to complete explicitly before moving on to parts (c) and (d).

Though students are encouraged to come up with multiple solution techniques, the solution targeted by the series of prompts in the task is to start with the biggest unit fraction less than $\frac{1}{n}$, namely $\frac{1}{n+1}$. This leads to the expression

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n^2+n},$$

an algebraic identity of rational expressions. Teachers looking to increase the cognitive complexity of the task might withhold the given identity, and have students search for some on their own. Alternatives solution techniques abound: Another good method is to use the identity

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6},$$

to deduce (upon dividing by n) that

$$\frac{1}{n} = \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n}.$$

A solution of the task using this identity instead is addressed in a second solution.

As optional scaffolding, the instructor may wish to encourage students to use part (c) to help with part (d), as is done in the solutions.

Students working on this problem will engage in several of the standards for mathematical practice:

- MP1: Make Sense of Problems and Persevere in Solving Them. Students will need to understand what an Egyptian fraction is and how to convert a fraction to this form. This will require patience and experimentation as we are not used to expressing fractions this way.
- MP2: Reason Abstractly and Quantitatively. Students will need to make fraction conversions both in a concrete and abstract situation.
- MP7: Look For and Make Use of Structure. In order to find an egyptian fraction representation for a general fraction $\frac{p}{q}$ the students will need to identify a pattern and also explain how to implement their strategy which will require repeated use of a common argument.
- MP8: Look For and Express Regularity in Repeated Reasoning. Part c of the first solution and the second solution both apply an iterative process to write a fraction as an Egyptian fraction.

Some additional questions the teacher may wish to address or which may come up in working on this task are:

a. Is there an algorithm for finding an Egyptian fraction for a given number? For numbers like $\frac{3}{7}$ and $\frac{5}{13}$ it is possible to find an egyptian fraction representation through trial and error. For a number like $\frac{357}{1123}$ this will be much more challenging.

b. Can you determine the smallest number of unit fractions needed in an Egyptian fraction representation of a given number?

c. One remarkable consequence of part (d) of this problem is that as m grows, the sum

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}$$

eventually becomes larger than any given number. For example, using part (b) we can write 1,000,000 or 1,000,000,000 or any other number as an Egyptian fraction so when

m is big enough, the sum of all of the unit fractions with denominators at most m can be made to exceed 1,000,000 or 1,000,000,000 or any other positive number.

To view this task as part of the progression of learning inherent in CCSSM, teachers may wish to observe relations with analogous content in the 5th grade, e.g., found in the task 5-NF Egyptian Fractions.

Solutions

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Solution: 1

a. There are many strategies for doing this, and indeed many ways of writing a fraction as an Egyptian fraction. Though we'll pursue one particular line of thought in subsequent parts, we'll address a few options here. For $\frac{2}{7}$, one way to write this as a sum of unit fractions would be to note that $\frac{1}{4} < \frac{2}{7}$ and it is the biggest unit fraction less than $\frac{2}{7}$. Subtracting it from $\frac{2}{7}$, we find that

$$\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$$

and this is an Egyptian fraction representation of $\frac{2}{7}$.

We can employ the same method for $\frac{5}{7}$, which is just larger than $\frac{1}{2}$, the largest of the unit fractions. We have

$$\frac{5}{7} = \frac{1}{2} + \frac{3}{14}.$$

We now have to deal with $\frac{3}{14}$ which lies between $\frac{1}{5}$ and $\frac{1}{4}$. We have

$$\frac{3}{14} = \frac{1}{5} + \frac{1}{70}.$$

Putting this together with our previous equation gives an Egyptian fraction representation of $\frac{5}{7}$:

$$\frac{5}{7} = \frac{1}{2} + \frac{1}{5} + \frac{1}{70}.$$

Starting over with $\frac{2}{7}$, an alternative idea is to write $\frac{2}{7}$ as $\frac{2}{7} = \frac{1}{7} + \frac{1}{7}$. Now we have to break down one of the $\frac{1}{7}$'s and we could do this by writing

$$\frac{1}{7} = \frac{1}{8} + \frac{1}{56},$$

foreshadowing the upcoming identity.

Putting these two expressions together gives

$$\frac{2}{7} = \frac{1}{7} + \frac{1}{8} + \frac{1}{56}$$

Similarly, for $\frac{5}{13}$ we could also start by writing this as

$$\frac{5}{13} = \frac{1}{13} + \frac{4}{13}.$$

For $\frac{4}{13}$ this is a little less than a third and more than one fourth. We have

$$\frac{4}{13} = \frac{1}{4} + \frac{3}{52}.$$

For $\frac{3}{52}$ this is a little more than $\frac{1}{17}$ and less than $\frac{1}{18}$. We find

$$\frac{3}{52} = \frac{1}{18} + \frac{1}{468}$$

Putting these calculations together gives

$$\frac{5}{13} = \frac{1}{13} + \frac{1}{4} + \frac{1}{18} + \frac{1}{468}.$$

These numbers get large very quickly and so a good, uniform method for how to proceed is definitely needed.

b. One way to verify the identity is beginning with the identity $x + 1 = x + 1$ for any x, and then dividing both sides by $x(x + 1)$ (valid since x is positive, and so this term is not zero) and distributing. This gives

$$\frac{x+1}{x(x+1)} = \frac{x}{x(x+1)} + \frac{1}{x(x+1)}.$$

After cancelling common terms, we're left with the desired identity. A second derivation is noted in the next part of the solution.

c. To write $\frac{1}{n}$ as a sum of smaller unit fractions, we need only apply the previous identity. Note that, mirroring part (a), we can view this identity as beginning the Egyptian fraction representation with the largest unit fraction smaller than $\frac{1}{n}$, namely, $\frac{1}{n+1}$. We then have

$$\begin{aligned}\frac{1}{n} - \frac{1}{n+1} &= \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} \\ &= \frac{1}{n(n+1)},\end{aligned}$$

which also provides an alternative derivation of the identity in part (b).

Rewriting this we get

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$$

which is an alternative Egyptian fraction representation of $\frac{1}{n}$.

d. We can write

$$\frac{p}{q} = \frac{1}{q} + \ldots + \frac{1}{q}$$

where there are p copies of the unit fraction $\frac{1}{q}$. If $p = 1$ then this is an Egyptian fraction and we are done. If not, we can keep the first $\frac{1}{q}$ and then replace the second one with $\frac{1}{q+1} + \frac{1}{q(q+1)}$. If $p = 2$ then we are done, having expressed $\frac{p}{q}$ as a sum of three unit fractions. If $p > 2$ then we can write the third $\frac{1}{q}$ as $\frac{1}{q+1} + \frac{1}{q(q+1)}$ and then replace each of these using the same idea: so $\frac{1}{q+1}$ could be written as

$$\frac{1}{q+1} = \frac{1}{q+2} + \frac{1}{(q+1)(q+2)}$$

and $\frac{1}{q(q+1)}$ can be replaced with

$$\frac{1}{q(q+1)} = \frac{1}{q(q+1)+1} + \frac{1}{q(q+1)(q(q+1)+1)}.$$

These expressions get very complicated very quickly but the important thing to note is that since q is at least 2 the denominators of these unit fractions are getting bigger and bigger. So if we continue to repeat this procedure, replacing every duplicate unit fraction with two unit fractions with larger denominators, then we will eventually remove all duplicates and have a (very complex) Egyptian fraction representation for $\frac{p}{q}$.

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Solution: 2

For the curious, we present the solutions to part (c) and (d) which make use of the remark in the commentary that a second solution stems from the identity

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

Multiplying both sides by $\frac{1}{n}$ gives

$$\frac{1}{n} = \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n},$$

so we have represented $\frac{1}{n}$ as a sum of three different unit fractions. For an arbitrary $\frac{p}{q}$, we begin by writing

$$\frac{p}{q} = \frac{1}{q} + \cdots + \frac{1}{q}$$

where there are p terms in the sum. We can only have one $\frac{1}{q}$ in the expression so we may take the second $\frac{1}{q}$ and replace it with

$$\frac{1}{2q} + \frac{1}{3q} + \frac{1}{6q}.$$

We now move to the third copy of $\frac{1}{q}$ and replace it with

$$\frac{1}{2q} + \frac{1}{3q} + \frac{1}{6q}.$$

Each of these unit fractions has already been used so these also need to be replaced.

Since each replacement makes the denominators bigger, we will be able to eventually get all different unit fractions in our expansion, that is we will eventually get an expression of $\frac{p}{q}$ as an Egyptian fraction.

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